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CPN models in general coordinates

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Abstract

An analysis of CPN models is given in terms of general coordinates or arbitrary interpolating fields. Only closed expressions made from simple functions are involved. Special attention is given to CP2 and CP4. In the first of these the retrieval of stereographic coordinates reveals the Hermitian form of the metric. A similar analysis for the latter case allows comparison with the Fubini–Study metric.

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1. Introduction

Despite the central importance of CPN models [1], and their recent revival as arising in supersymmetric form from minimized linear models [2] following the revision of the underlying supersymmetry algebra of densities to include central terms, there appears to be no treatment of them in general coordinates which would allow arbitrary field redefinitions for the interpolating Goldstone bosons. In this paper just such an analysis is presented. The next section explains how this is achieved by embedding the necessary structure into a more complicated one. Strangely, perhaps, nothing is needed but simple functions, and a completely general solution is found in closed form. In the following section the special case of CP2 and stereographic coordinates is presented. Then the corresponding step is made for CP4 allowing the connection to the Fubini–Study metric. Finally, there are brief conclusions and suggestions are made for future work.

2. General framework

Curiously this section begins by consideration of the embedding of the structure needed for the current problem into that of a larger system which has previously been solved in general coordinates leading to a closed form involving only simple functions [3]. The embedding is unique. Thus the starting point is a review of this established larger system and its solution, in which the liberty of changing notation (slightly) for convenience has been taken.

Consider then the Lie algebra of SU_n specified by taking as a basis the set of $(n^2 - 1)$ traceless Hermitian $n \times n$ matrices λ_i with the product law

$$\lambda_i \lambda_j = \frac{2}{n}\delta_{ij} + (d_{ijk} + \mathrm{i} f_{ijk})\lambda_k \tag{1}$$

as specified by Gell-Mann and Ne'eman [4]. When specific values of the structure constants f_{ijk}, or the symmetric d_{ijk} tensors or the λ_i matrices themselves are needed then the notation of [4] will be assumed. An element of the group SU_n, $g(\theta)$ is specified in exponential form by a set of $(n^2 - 1)$ real parameters θ_i, so that in infinitesimal form the transformations

$$q_A \longrightarrow q_A - \frac{\mathrm{i}}{2}\theta^i (\lambda_i)_{AB} q_B \tag{2}$$

and

$$M_i \longrightarrow M_i + \theta_k f_{ikj} M_k \tag{3}$$

specify the behaviour of the basic spinor q_A (quark) and adjoint vector M_i fields. Now define a traceless matrix M by

$$M_{KL} = M_i (\lambda_i)_{KL} \tag{4}$$

so that

$$M_i = \tfrac{1}{2}\,\mathrm{Tr}(M\lambda_i) \tag{5}$$

then a group element $g(\theta)$ which induces a unitary transformation

$$q_A \longrightarrow U(\theta)_{AB} q_B \tag{6}$$

on the basic spinors clearly induces an orthogonal transformation

$$M_i \longrightarrow R_{ij} M_j = \tfrac{1}{2}\,\mathrm{Tr}(U^{-1}\lambda_i U \lambda_j) M_j \tag{7}$$

on the adjoint representation.

The algebra of $SU_n \times SU_n$ is spanned by two sets of $(n^2 - 1)$ orthogonal elements L_i and R_i satisfying the commutation relations

$$[L_i, L_j] = \mathrm{i} f_{ijk} L_k \tag{8}$$

$$[R_i, R_j] = \mathrm{i} f_{ijk} R_k \tag{9}$$

$$[L_i, R_j] = 0 \tag{10}$$

and the linear combinations

$$V_i = L_i + R_i \tag{11}$$

$$A_i = L_i - R_i \tag{12}$$

are frequently used. Obviously the V_i generate a SU_n subgroup which is parity conserving. An element of the $SU_n \times SU_n$ group may be specified by two sets of $(n^2 - 1)$ real parameters, and the alternative expressions

$$g = \exp(-\mathrm{i}[\theta_i^V V_i + \theta_i^A A_i]) \tag{13}$$

and

$$g = \exp(-\mathrm{i}\theta_i^L L_i)\exp(-\mathrm{i}\theta_i^R R_i) \tag{14}$$

will prove useful with

$$\theta_i^L = \theta_i^V + \theta_i^A \tag{15}$$

$$\theta_i^R = \theta_i^V - \theta_i^A \tag{16}$$

specifying the correspondence. Every element of the group can also be decomposed into a product of the form

$$g = \exp(-\mathrm{i}\phi_i A^i)\exp(-\mathrm{i}\theta_i V^i) \tag{17}$$

which is unique in a neighbourhood of the identity element and this will play a crucial role in the general nonlinear realization scheme. The linear transformation laws are best specified by giving the quarks a Dirac spinor index in the usual manner and taking

$$q \longrightarrow q - \frac{\mathrm{i}}{2}\theta_i^L \lambda_i \frac{(1+\gamma_5)}{2} q - \frac{\mathrm{i}}{2}\theta_i^R \lambda^i \frac{(1-\gamma_5)}{2} q \tag{18}$$

as the concrete infinitesimal form.

Since the matrices

$$P_L = \frac{(1+\gamma_5)}{2} \tag{19}$$

and

$$P_R = \frac{(1-\gamma_5)}{2} \tag{20}$$

act as a standard set of projection operators, the treatment of linearly transforming multiplets of $SU_n \times SU_n$ now follows trivially.

To treat the nonlinear realizations of $SU_n \times SU_n$ in full generality the (n^2-1) Hermitian components M_i of the adjoint vector of SU_n must be considered in more detail. In the terminology of Michel and Radicati [5], the vector is said to be generic (or to belong to the generic stratum) if all eigenvalues of M are distinct. For the generic case the minimal polynomial for the matrix is the characteristic polynomial satisfying the equation

$$\prod_{A=1}^{n}(M - m_A) = 0 \tag{21}$$

where the m_A are the eigenvalues which satisfy

$$\sum_{A=1}^{n} m_A = 0 \tag{22}$$

if the matrix is traceless. Thus the $(n-1)$ vectors with components given by powers of the matrix in the form

$$M^{\alpha}{}_i = \tfrac{1}{2}\,\mathrm{Tr}([M]^{\alpha}\lambda_i) \qquad [\alpha = 1, 2, \ldots, (n-1)] \tag{23}$$

are a linearly independent set, and the quantities

$$S_A = \mathrm{Tr}([M]^A) = \sum_{B=1}^{n}[m_B]^A \equiv \sum_{B=1}^{n} m_{AB} \tag{24}$$

are $(n-1)$ independent SU_n invariants (S_1 is identically zero). At once it is clear that the general vector which can be constructed from the M_i has the form

$$\xi_i = F_\alpha M_{\alpha i} \tag{25}$$

where the F_α are functions of the $(n-1)$ independent SU_n invariants. This freedom has been discussed at length by Gasiorowicz and Geffen [6]. From the point of view of field theory it corresponds to freedom of choice of interpolating fields. Provided that $F_1(0)$ is taken to be unity, and parity is respected, then all ξ_i so defined are equally good interpolating fields. From a geometrical viewpoint the ξ_i may be regarded as coordinates of points of the (n^2-1)-dimensional coset space manifold formed by the quotient of $SU_n \times SU_n$ by the vector

SU_n subgroup. The freedom is then viewed as the ability to change coordinates within a local patch near the origin.

An arbitrary point on the manifold is parameterized by

$$\exp(-\mathrm{i}\xi_i A_i) \equiv L(M) \equiv \exp\left(\frac{-\mathrm{i}\theta}{2\phi} M_i \lambda_i [P_L - P_R]\right) \tag{26}$$

where the first form corresponds with equation (17) and the second form represents the appropriate expression when equation (25) has been used so that the M_i are regarded as the coordinates, and $\phi^2 = M_i M_i$.

The general theory is well described by Coleman *et al* [7] and Callan *et al* [8], and the geometrical approach by Isham [9]. With the decomposition given in equation (17) the action of a general element g of the full group may be written as

$$g \exp(-\mathrm{i}\xi_i A_i) = \exp(-\mathrm{i}\xi_i' A_i) \exp(-\mathrm{i}\eta_i V_i) \tag{27}$$

$$\equiv L(M') \exp(-\mathrm{i}\eta_i V_i) \tag{28}$$

where M_i' and η_i both depend on M_i and g. Then the primary result of the general theory is that

$$g : M_i \longrightarrow M_i' \tag{29}$$

gives a nonlinear realization of the algebra which is linear on the SU_n vector subgroup. Moreover if h is an element of the vector subgroup and

$$h : \Psi_\Omega \longrightarrow D(h)_{\Omega\Gamma} \Psi_\Gamma \tag{30}$$

is a linear (unitary) representation of that subgroup, then

$$g : \Psi_\Omega \longrightarrow D[\exp(-\mathrm{i}\eta_i V_i)]_{\Omega\Gamma} \Psi_\Gamma \tag{31}$$

gives a realization of the full group. Notice that this latter transformation is linear in Ψ but nonlinear (through η_i) in the M_i when g is not in the vector subgroup. Fields which transform according to equation (31) are called standard fields, and it is important to understand that by a suitable redefinition of coordinates any nonlinear realization of $SU_n \times SU_n$ which is linear on the vector subgroup can be brought into this standard form. In practice the most useful result is that, if one has a linear irreducible (unitary) representation of $SU_n \times SU_n$ such that

$$g : N_\Omega \longrightarrow D[g]_{\Omega\Gamma} N_\Gamma \tag{32}$$

then

$$\Psi_\Omega(M) = D[L^{-1}(M)]_{\Omega\Gamma} N_\Gamma \tag{33}$$

transform as the components of standard fields.

It is now clear that there are just three classes of fields to consider:

(1) Linear representations which may be built up in the usual way as multispinors with transformation laws defined by equation (18). These will not be treated in more detail.

(2) Vectors M_i transforming as the adjoint representation of SU_n with a nonlinear transformation law under chiral action specified by equation (27). These will allow a description of the massless Goldstone bosons (pions etc) corresponding to the axial degrees of freedom spontaneously violated. The specification of invariants constructed (nonlinearly) from these is most important and will be exhibited later.

(3) Standard fields which appear linearly in their transformation laws, but with nonlinear functions of the M_i induced according to equations (31) and (28). These are important in describing matter (e.g. nucleons) interacting with the Goldstone bosons as chiral matter. Once more, the specification of the corresponding invariants is most important and will be given later.

The technical problem of finding the invariants is solved in [3]. A crucial step is the resolution of the powers of the matrix M in the form

$$[M]^A = [m_B]^A P_B \equiv m_{AB} P_B \tag{34}$$

where the P_B are n Hermitian matrices, each $n \times n$, with the properties

$$P_A P_B = \delta_{AB} P_B \text{ (no sum)} \tag{35}$$

$$\mathrm{Tr}(P_A) = 1 \tag{36}$$

and

$$\sum_{A=1}^{n} P_A = 1 \tag{37}$$

where this 1 is the unit $(n \times n)$ matrix. Although the P_A are not in general diagonal, the above projection operator properties make calculations tractable. Now define

$$P_{Ai} = \tfrac{1}{2}\,\mathrm{Tr}(P_A \lambda_i) \tag{38}$$

and

$$(P_A)_{MN} = P_{Ai}(\lambda_i)_{MN} + \frac{1}{n}\delta_{MN} \tag{39}$$

where because the P_A are complete it follows that

$$\sum_{A=1}^{n} P_{Ai} = 0 \tag{40}$$

and, introducing

$$p_{Ai} = \sqrt{2}[P_{Ai} - (1 + \sqrt{n})^{-1}]P_{ni} \tag{41}$$

with

$$\sqrt{2}P_{Ai} = p_{Ai} + \frac{1}{\sqrt{n}}p_{ni} \tag{42}$$

establishes that $p_{\mu i}$ for $\mu = 1, 2, \ldots, (n-1)$ are orthonormal.

The second-rank tensors defined by the M_i are conveniently handled by an extension of these ideas, and fall into two classes. One such class is formed by the $n(n-1)$ independent tensors defined by

$$(P_{AB})_{ij} \equiv P_{AiBj} \equiv \tfrac{1}{2}\,\mathrm{Tr}(P_A \lambda_i P_B \lambda_j) \qquad (A \neq B) \tag{43}$$

and

$$I_{ij} = \tfrac{1}{2}\,\mathrm{Tr}(P_A \lambda_i P_A \lambda_j), \tag{44}$$

which have the properties

$$II = I \tag{45}$$

$$IP_{AB} = 0 = P_{AB} I \tag{46}$$

and

$$P_{AB} P_{CD} = \delta_{AC} \delta_{BD} P_{AB} \text{ (no sum)} \tag{47}$$

in terms of the matrix notation of the last section. Moreover, these are all Hermitian matrices and the trace of each P_{AB} is unity. Since it is easy to show also that

$$\sum_{A \neq B}{}' P_{AB} = 1 - I \tag{48}$$

where the sum is over all A and B but excluding terms with $A = B$, this gives a projection operator resolution in one sector of the space of these second-rank tensors and so I (with trace $[n-1]$) will decompose further. The second class of tensors may be identified with the $(n-1)^2$ independent matrices with components

$$(p_{\alpha\beta})_{ij} \equiv p_{\alpha i} p_{\beta j} \tag{49}$$

which span the subspace of $(n^2-1) \times (n^2-1)$ matrices projected out on multiplication by I from both sides and which are therefore orthogonal to the subspace in which the P_{AB} lie. Since the $p_{\alpha i}$ are orthonormal, the multiplication law for the $p_{\alpha\beta}$ is

$$p_{\alpha\beta} p_{\gamma\delta} = \delta_{\beta\gamma} p_{\alpha\delta}. \tag{50}$$

It has been established by Barnes and Delbourgo [10] that all the independent second-rank tensors which can be constructed from the M_i are spanned by the $(n-1)(2n-1)$ independent $p_{\alpha\beta}$ and P_{AB}.

The most general unitary unimodular matrix U constructed from the M_i may be written in the form

$$U = U_A P_A = \exp\left[\frac{-\mathrm{i}}{2}\theta_A\right] P_A \qquad \text{where} \quad \sum_{A=1}^{n} \theta_A = 0 \tag{51}$$

but the θ_A are otherwise completely arbitrary independent functions of the independent SU_n invariants S_A subject to the considerations of parity and weak field limits as mentioned before. These $(n-1)$ effective arbitrary functions of the $(n-1)$ invariants are characteristic of the general solution and will persist throughout this work.

It has been conventional to define

$$\sqrt{2}\phi_A = m_A - (1+\sqrt{n})^{-1} m_n \tag{52}$$

with

$$m_A = \sqrt{2}(\phi_A + n^{-\frac{1}{2}}\phi_n) \tag{53}$$

so that, extending the notation used previously,

$$M_i = \phi_\alpha p_{\alpha i} \tag{54}$$

and

$$\phi_{\alpha,i} = p_{\alpha i} \tag{55}$$

follow immediately. Similarly, defining

$$\sqrt{2}\psi_A = \theta_A - (1+n^{\frac{1}{2}})^{-1}\theta_n \tag{56}$$

with

$$\theta_A = \sqrt{2}(\psi_A + n^{-\frac{1}{2}}\psi_n) \tag{57}$$

the ψ_α may be treated as $(n-1)$ independent (arbitrary) functions of the ϕ_α which then serve as the $(n-1)$ independent invariants.

The transformation laws for all realizations are now given in [3] in closed form and in terms of simple functions. Restricting attention to first-order derivatives of the fields with respect to space and time, and also restricting attention to a study of the Goldstone boson fields M_i and the standard fields the results can be given in terms of the general analysis of [7] and [8]. There are two important results. First, although $\partial_\mu M_i$ and $\partial_\mu \Psi_\Gamma$ do not transform as standard fields, the covariant derivatives

$$D_\mu M_i = a_{\mu i} \tag{58}$$

and

$$D_\mu \Psi_\Gamma = \partial_\mu \Psi_\Gamma - \mathrm{i}\nu_{\mu i}(T_i)_{\Gamma\Omega}\Psi_\Omega \tag{59}$$

where under SU_n

$$\Psi_\Gamma \longrightarrow \Psi_\Gamma - \mathrm{i}\theta_i(T_i)_{\Gamma\Omega}\Psi_\Omega \tag{60}$$

and where

$$L^{-1}(M)\partial_\mu L(M) = \exp(-\xi_i A_i)\partial_\mu \exp(\xi_j A_j) = \nu^i_\mu V_i + a^i_\mu A_i \tag{61}$$

have precisely this property. Secondly, they show that the most general Lagrangian of the type under consideration may be written as a function of the standard fields Ψ, $D_\mu\Psi$ and $D_\mu M_i$ only; that is the M_i will not appear explicitly, and the Goldstone bosons will be massless. It then follows that the Lagrangian so formed will be invariant under $SU_n \times SU_n$ if and only if it is constructed to be invariant under the SU_n vector subgroup. This latter requirement is, of course, achieved by index saturation once more.

The result given in [3] (now dropping the chiral projectors and normalizing for this problem) takes the concrete form

$$D_\mu M_i = \left\{\frac{\partial\psi_\beta}{\partial\phi_\gamma}(P_{\gamma\beta})_{ik} + \sum_{A\neq B}{}' \frac{\sqrt{2}}{(\phi_A - \phi_B)} \sin\left[\frac{\psi_A - \psi_B}{\sqrt{2}}\right](P_{AB} + P_{BA})_{ik}\right\}(\partial_\mu M_k) \tag{62}$$

and represents a complete specification of the required Lagrangian in simple closed form. Using the geometric formulation of Isham [9] gives the coset space metric in the form related to the covariant derivatives as

$$g_{ij}(\partial_\mu M_i)(\partial^\mu M_j) = (D_\mu M_i)(D^\mu M_i) \tag{63}$$

and we have normalized g_{ij} to δ_{ij} in the limit of zero fields. In matrix notation this yields

$$g = \frac{1}{4}\left\{p_{\beta\lambda}\frac{\partial\psi_\alpha}{\partial\phi_\beta}\frac{\partial\psi_\alpha}{\partial\phi_\lambda} + \sum_{A\neq B}{}' \frac{2}{(\phi_A - \phi_B)^2}(P_{AB} + P_{BA})\sin^2\left[\frac{\psi_A - \psi_B}{\sqrt{2}}\right]\right\} \tag{64}$$

immediately because of the orthonormality.

At last it is time to see how this structure is related to CP$_n$. Returning to the $SU_n \times SU_n$ action given in equations (27) and (28), consider the restriction of ξ_i to the subset of dimension $2(n-1)$ given by

$$A_{(n-1)^2}, A_{(n-1)^2+1}, \ldots, A_{n^2-2} \tag{65}$$

and similarly the restriction of V_i to the subset of dimension $(n-1)^2$ given by

$$V_1, V_2, \ldots, V_{(n-1)^2-1} = V_{n(n-2)} \qquad \text{and} \qquad V_{n^2-1} = V_{(n+1)(n+1-2)} \tag{66}$$

the restrictions are overall obviously unique. The remaining V_i after the restriction clearly generate $SU_{n-1} \times U_1$, and the remaining A_i combine with the V_i to yield the whole SU_n in which the former are uniquely embedded. This gives the manifold $SU_n/(SU_{n-1} \times U_1)$ which is of dimension $2(n-1)$ and forms the basis for CP$_2(n-1)$. All the previous results now apply to this embedded space simply by applying the same restrictions.

It is still necessary to interpret the information thus obtained in terms of the CP$_n$ structure. From this viewpoint the $V_1, V_2, \ldots, V_{n^2-1}$ and $V_{n(n+2)}$ generate an $SU_{n-1} \times U_1$ under which the A_μ transform linearly as a complex $2(n-1)$ dimensional multiplet.

Recall that the $2(n-1)\xi_i$ are the generalized coordinates or interpolating fields for the massless Goldstone bosons. We can combine these into $(n-1)$ complex coordinates by taking

$$z_0 = \xi_0 - \mathrm{i}\xi_1, \qquad z_1 = \xi_2 - \mathrm{i}\xi_3, \ldots, z_{n-2} = \xi_{2(n-1)} - \mathrm{i}\xi_{2(n-1)+1}$$

by a judicious choice of labels.

We can see that with the new labels then

$$M = \sum_{\mu=0}^{n-2} \left[z_\mu \frac{[\lambda_{2\mu+(n-1)^2} + \mathrm{i}\lambda_{2\mu+(n-1)^2+1}]}{2} + \bar{z}_\mu \frac{[\lambda_{2\mu+(n-1)^2} - \mathrm{i}\lambda_{2\mu+(n-1)^2+1}]}{2} \right]$$

having only non-zero entries going from z_0 to z_{n-2} down the final right-hand column from the top, and going from $\bar{z}_0$ to $\bar{z}_{n-2}$ across the final row from the left. There are zeros in the top $(n-3) \times (n-3)$ left-hand block, and a zero in the bottom right-hand corner. Each of the complex z's gives two vectors in the coset space. The corresponding lengths can be expressed in terms of the independent $(n-2)$ invariants ϕ_α out of which the $(n-2)$ independent functions θ_α (used in constructing the z's) are formed.

3. The special cases of CP2 and CP4

The CP2 case has previously been called the chiral 2-sphere by Barnes *et al* [11] when it has been described in some detail. In the present notation M takes the form

$$M = \tfrac{1}{2}(z + \bar{z})\sigma_1 + \tfrac{1}{2}\mathrm{i}(z - \bar{z})\sigma_2 = M_A \sigma_A \tag{67}$$

where z_0 is written as z and where $\phi^2 = z\bar{z}$, and writing $M_A = \phi n_A$ gives

$$(P_{12} + P_{21})_{AB} = \delta_{AB} - n_A n_B. \tag{68}$$

Thus putting

$$\psi_1 - \psi_2 = \sqrt{2}\theta \tag{69}$$

and

$$\phi_1 - \phi_2 = \sqrt{2}\phi \tag{70}$$

one finds immediately that

$$g_{AB} = \frac{1}{4} \left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi} \right)^2 n_A n_B + \frac{\sin^2\theta}{\phi^2} (\delta_{AB} - n_A n_B) \right]. \tag{71}$$

Note that in this example where there is only a single arbitrary function θ of a single invariant ϕ, the notation of the δSU_2 description does not need adapting for the SU_2/U_1 coset space.

The condition to find the Hermitian form is obviously

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\phi} \right)^2 = \frac{\sin^2\theta}{\phi^2} \tag{72}$$

with the solution

$$\phi = c \tan\left(\frac{\theta}{2}\right) \tag{73}$$

where c is a constant, being the one conventionally chosen. This is the coordinate system usually known as stereographic. Obviously equation (71) now yields

$$g_{AB} = \frac{\delta_{AB}}{[1 + z\bar{z}]} \tag{74}$$

where

$$z_0 = M_1 + \mathrm{i}M_2 \qquad \text{when} \quad c = 1 \tag{75}$$

and hence it follows that

$$\mathcal{L}_2 = \frac{1}{2} g_{AB}(\partial_\mu M_A)(\partial^\mu M_B) = \frac{1}{[1 + z_0\bar{z}_0]} \frac{(\partial_\mu z_0)(\partial^\mu \bar{z}_0)}{2} \tag{76}$$

in obvious Hermitian form in these stereographic coordinates, and sometimes this is written as

$$\mathcal{L}_2 = \frac{(\partial_\mu \xi)(\partial^\mu \bar{\xi})}{2[r^2 + \xi\bar{\xi}]} \tag{77}$$

where $rz_0 = \xi$ is used to emphasize the constant radius r of the 2-sphere.

The CP4 case has

$$M = M_\mu \lambda_\mu = \frac{(z_0 + \bar{z}_0)}{2}\lambda_4 + \mathrm{i}\frac{(z_0 - \bar{z}_0)}{2}\lambda_5 + \frac{(z_1 + \bar{z}_1)}{2}\lambda_6 + \mathrm{i}\frac{(z_1 - \bar{z}_1)}{2}\lambda_7. \tag{78}$$

This is perhaps a suitable place to note that if the functions z_0 and z_1 are not chosen carefully then M will not be generic and the degree of the equation satisfied by it will be less than the maximum.

The Goldstones bosons of this scheme are the octet of pseudo scalar mesons described by the M_i. In general there are two SU_3 invariants which may be constructed from the M^i. These can be denoted

$$X = M^i M_i \tag{79}$$

and

$$Y = d_{ijk} M^i M^j M^k \tag{80}$$

where the determinantal inequality

$$3Y^2 \leqslant X^3 \tag{81}$$

ensures that the norm of an arbitrary vector constructed from the M^i shall be positive definite. Now define ϕ and δ by

$$\phi = X^{\frac{1}{2}} \tag{82}$$

and

$$\phi^3 \sin\delta = \sqrt{3}Y \tag{83}$$

as the basic invariants.

It is straightforward to show [12] that, if

$$N_i = d_{ijk} M^j M^k \tag{84}$$

then

$$\hat{m}_i = \phi^{-1} M_i \tag{85}$$

and

$$\hat{r}_i = \phi^{-2} \sec\delta(\sqrt{3}N_i - \phi M^i \sin\delta) \tag{86}$$

are an orthonormal base for the independent vectors.

It has also been shown that the vectors

$$q_i = \hat{r}_i \cos\alpha + \hat{m}_i \sin\alpha \tag{87}$$

and

$$s_i = (-)\hat{r}_i \sin\alpha + \hat{m}_i \cos\alpha \tag{88}$$

with

$$3\alpha = \delta - 2A\pi \qquad (A = 1, 2, 3) \tag{89}$$

are respectively charge and special vectors in the sense of Michel and Radicati [5]. Apart from their orthonormality these vectors also have the properties

$$(-)\sqrt{3}d_{ijk}q^j q^k = q_i = \sqrt{3}d_{ijk}s^j s^k \tag{90}$$

$$\sqrt{3}d_{ijk}s^j q^k = s_i \tag{91}$$

and

$$f_{ijk}s^j q^k = 0 \tag{92}$$

so that a single pair q^i and s^i represent a useful alternative to working with the three P_i^A which are linearly dependent. Adopting the choice $A = 3$ for the set of standard q^i and s^i it follows that

$$\sqrt{2}p_1^i = q_i + s_i \tag{93}$$

and

$$\sqrt{2}p_2^i = q_i - s_i \tag{94}$$

are the orthonormal basis vectors introduced previously.

The second rank tensors which may be constructed from the M^i are spanned by the six projection operators $(P_{AB})_{ij}$ and the four $(p_{\alpha\beta})_{ij}$, all of which are taken to be Hermitian in the matrix sense.

It is standard to introduce projection operators with a cyclic notation in the form

$$(S_1)_{ij} = (P_{23})_{ij} + (P_{32})_{ij} \tag{95}$$

$$(S_2)_{ij} = (P_{13})_{ij} + (P_{31})_{ij} \tag{96}$$

$$(S_3)_{ij} = (P_{12})_{ij} + (P_{21})_{ij}. \tag{97}$$

The first term in equation (64) can be treated by making the substitutions (where lower and upper Greek indices take the ranges 4–5 and 6–7 respectively)

$$p_1^\mu \Rightarrow n_1^\mu, \qquad p_2^\Gamma \Rightarrow n_2^\Gamma \tag{98}$$

$$\frac{\partial\psi^\beta}{\partial_\mu\phi_1} \Rightarrow \sqrt{2}\frac{\partial\psi^\beta}{\partial_\mu\omega_1} = \sqrt{2}\frac{\partial\psi^\beta}{\partial\omega_1}\frac{\partial\omega_1}{\partial M_\mu} = \sqrt{2}\frac{\partial\psi^\beta}{\partial\omega_1}\frac{M_\mu}{\omega_1} \tag{99}$$

$$\frac{\partial\psi^\beta}{\partial_\Gamma\phi_2} \Rightarrow \sqrt{2}\frac{\partial\psi^\beta}{\partial_\Gamma\omega_2} = \sqrt{2}\frac{\partial\psi^\beta}{\partial\omega_2}\frac{\partial\omega_2}{\partial M_\Gamma} = \sqrt{2}\frac{\partial\psi^\beta}{\partial\omega_2}\frac{M_\Gamma}{\omega_2} \tag{100}$$

and similarly, using equations (95)–(97), the S_A can be brought to the forms

$$(S_1)_{\mu\nu} = \delta_{\mu\nu} - n'_\mu n'_\nu \tag{101}$$

$$(S_2)_{\Gamma\Omega} = \delta_{\Gamma\Omega} - n_\Gamma^2 n_\Omega^2 \tag{102}$$

and S_3 vanishes because n^3 lies inside the $SU_2 \times U_1$ subspace rather than in the coset space. (This explains why the range of summation is reduced in future.)

It follows that

$$4g_{\mu\nu} = 2\frac{\partial\psi^\beta}{\partial\omega_1}\frac{\partial\psi^\beta}{\partial\omega_1}\frac{M_\mu M_\nu}{(\omega_1)^2} + \frac{4(S_1)_{\mu\nu}}{(\omega_1)^2}\sin^2\left(\frac{\theta_1 + 2\theta_2}{2}\right) \tag{103}$$

$$4g_{\Gamma\Omega} = 2\frac{\partial\psi^\beta}{\partial\omega_2}\frac{\partial\psi^\beta}{\partial\omega_2}\frac{M_\Gamma M_\Omega}{(\omega_1)^2} + \frac{4(S_2)_{\Gamma\Omega}}{(\omega_2)^2}\sin^2\left(\frac{\theta_2 - 2\theta_1}{2}\right) \tag{104}$$

and

$$4g_{\mu\Omega} = 2\frac{\partial\psi^\beta}{\partial\omega_1}\frac{\partial\psi^\beta}{\partial\omega_2}\frac{M_\mu M_\Omega}{\omega_1\omega_2} \tag{105}$$

Noting that $M_\mu \equiv \omega_1 n^1_\mu$, it appears that the hermiticity conditions on the diagonal components are

$$\frac{\partial\psi^\beta}{\partial\omega_1}\frac{\partial\psi^\beta}{\partial\omega_1} = \frac{2}{(\omega_1)^2}\sin^2\left(\frac{\theta_1+2\theta_2}{2}\right) \tag{106}$$

and

$$\frac{\partial\psi^\beta}{\partial\omega_2}\frac{\partial\psi^\beta}{\partial\omega_2} = \frac{2}{(\omega_2)^2}\sin^2\left(\frac{\theta_2-2\theta_1}{2}\right). \tag{107}$$

These conditions can be imposed by a slight generalization of the method used in the CP2 case. Obviously it will be advantageous to introduce the abbreviations

$$D = \frac{\partial\psi_1}{\partial\omega_1}, \qquad D' = \frac{\partial\psi_2}{\partial\omega_1}, \qquad d = \frac{\partial\psi_1}{\partial\omega_2} \qquad \text{and} \qquad d' = \frac{\partial\psi_2}{\partial\omega_2} \tag{108}$$

It is simple to see that from equations (106) and (107) it follows that

$$D^2 + D'^2 = \frac{2}{(\omega_1)^2}\sin^2\left[\frac{\theta_1+2\theta_2}{2}\right] \tag{109}$$

and

$$d^2 + d'^2 = \frac{2}{(\omega_2)^2}\sin^2\left[\frac{\theta_2-2\theta_1}{2}\right]. \tag{110}$$

These two results ensure the hermiticity constraints on $g_{\mu\nu}$ and $g_{\Gamma\Omega}$, which then take forms

$$4g_{\mu\nu} = \frac{1}{(\omega_1)^2}\delta_{\mu\nu}\sin^2\left(\frac{\theta_1+2\theta_2}{2}\right) \tag{111}$$

and

$$4g_{\Gamma\Omega} = \frac{1}{(\omega_2)^2}\delta_{\Gamma\Omega}\sin^2\left(\frac{\theta_2-2\theta_1}{2}\right) \tag{112}$$

where the normalization has again been adjusted. The other components take the forms

$$4g_{\mu\Omega} = (Dd + D'd')\frac{n^1_\mu n^2_\Omega}{2} \tag{113}$$

and

$$4g_{\Gamma\nu} = (Dd + D'd')\frac{n^1_\nu n^2_\Gamma}{2} \tag{114}$$

and as these are off diagonal it is necessary to show hermiticity makes them zero.

Now put

$$\omega_1 = [c^2 + (\omega_2)^2]^{\frac{1}{2}}\tan\left(\frac{\theta_1+2\theta_2}{4}\right) \tag{115}$$

to show that

$$\frac{\partial\left(\frac{\theta_1+2\theta_2}{2}\right)}{\partial\omega_1} = \frac{\sin\left(\frac{\theta_1+2\theta_2}{2}\right)}{\omega_1} = \frac{2[c^2+(\omega_2)^2]^{\frac{1}{2}}}{[c^2+(\omega_1)^2+(\omega_2)^2]}. \tag{116}$$

Similarly, put

$$\omega_2 = [c'^2 + (\omega_1)^2]^{\frac{1}{2}}\tan\left(\frac{\theta_2-2\theta_1}{4}\right) \tag{117}$$

to show that

$$\frac{\partial\left(\frac{\theta_2-2\theta_1}{2}\right)}{\partial\omega_2}=\frac{\sin\left(\frac{\theta_2-2\theta_1}{2}\right)}{\omega_2}=\frac{2[c'^2+(\omega_1)^2]^{\frac{1}{2}}}{[c'^2+(\omega_1)^2+(\omega_2)^2]}. \tag{118}$$

It is straightforward to see from equations (106) and (107) that

$$\left(\frac{\partial\theta_1}{\partial\omega_1}\right)^2+\left(\frac{\partial\theta_1}{\partial\omega_1}\right)\left(\frac{\partial\theta_2}{\partial\omega_1}\right)+\left(\frac{\partial\theta_2}{\partial\omega_1}\right)^2=\frac{1}{(\omega_1)^2}\sin^2\left(\frac{\theta_1+2\theta_2}{2}\right) \tag{119}$$

and

$$\left(\frac{\partial\theta_1}{\partial\omega_2}\right)^2+\left(\frac{\partial\theta_1}{\partial\omega_2}\right)\left(\frac{\partial\theta_2}{\partial\omega_2}\right)+\left(\frac{\partial\theta_2}{\partial\omega_2}\right)^2=\frac{1}{(\omega_2)^2}\sin^2\left(\frac{\theta_2-2\theta_1}{2}\right) \tag{120}$$

so that, using the left-hand parts of equations (116) and (118), it follows that

$$c'=(\pm)c \tag{121}$$

is required since the c and c' are constants independent of the ω_1 and ω_2 variables, and the expressions on the right-hand sides result simply from using a trigonometric substitution to evaluate the integral. Hence, the forms

$$g_{\mu\nu}=\frac{\delta_{\mu\nu}[c^2+(\omega_2)^2]}{[c^2+(\omega_1)^2+(\omega_2)^2]^2} \tag{122}$$

and

$$g_{\Gamma\Omega}=\frac{\delta_{\Gamma\Omega}[c^2+(\omega_1)^2]}{[c^2+(\omega_1)^2+(\omega_2)^2]^2} \tag{123}$$

are revealed.

Now consider

$$D^2+D'^2=d^2+d'^2 \tag{124}$$

which follows from equations (109) and (110) by using the left-hand parts of equations (116) and (118), now that $c'=(\pm)c$, reveals that

$$\mathrm{d}D+\mathrm{d}'D'=0. \tag{125}$$

From equations (113) and (114), it is now evident that

$$g_{\mu\Omega}=0=g_{\Gamma\nu} \tag{126}$$

and this completes the specification of the metric through the hermiticity conditions. It is perhaps worth repeating that the forms of the metric (given for example in equations (103) and (104)) are in general coordinates before the hermiticity conditions are applied. However, the forms given in equations (122) and (123) are in Hermitian form and may be directly compared with the classic results of Fubini [13] and Study [14]. These authors apply scaling by using the conformal symmetry of the metric, and this is directly equivalent to setting $c = 1$ in the present notation. Reverting to complex notation reveals the invariant

$$\mathcal{L}_4=\frac{\mathrm{d}z_1\,\mathrm{d}\bar{z}_1[1+z_2\bar{z}_2]+\mathrm{d}z_2\,\mathrm{d}\bar{z}_2\,[1+z_1\bar{z}_1]}{[1+z_1\bar{z}_1+z_2\bar{z}_2]^2} \tag{127}$$

retrieving the Fubini–Study form.

4. Conclusions

It appears that the CPN metric has been found in general coordinates (in principle) for all N, and that in the cases of $N = 2$ and 4 well known forms are recovered in the Hermitian limit. Obviously, the algebraic effort required does rise with N but only simple functions ever appear.

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